

Massive Type II in Double Field Theory

Olaf Hohm and Seung Ki Kwak

*Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA
ohohm@mit.edu, sk_kwak@mit.edu*

Abstract

We provide an extension of the recently constructed double field theory formulation of the low-energy limits of type II strings, in which the RR fields can depend simultaneously on the 10-dimensional space-time coordinates and linearly on the dual winding coordinates. For the special case that only the RR one-form of type IIA carries such a dependence, we obtain the massive deformation of type IIA supergravity due to Romans. For T-dual configurations we obtain a ‘massive’ but non-covariant formulation of type IIB, in which the 10-dimensional diffeomorphism symmetry is deformed by the mass parameter.

1 Introduction

Double field theory is an approach to make T-duality manifest at the level of effective space-time theories by doubling the coordinates. It introduces the usual ‘momentum’ (space-time) coordinates x^i together with new ‘winding’ coordinates \tilde{x}_i and a covariant constraint that locally eliminates half of the coordinates. Originally accomplished for the bosonic string [1–3], more recently it has been extended to heterotic [4] and to type II superstrings [5, 6]. (For earlier and related work see [7–16].) In particular, the type II double field theory provides a unified description of the *massless* type IIA and type IIB theories. In this paper we will show that a minimal extension of this theory exists that also contains *massive* deformations [17]. (For earlier results on massive type IIA within ‘duality-covariant’ frameworks see [18–20].)

The massive extension of type IIA supergravity due to Romans [17] can be motivated as follows. If one introduces for each RR p -form the dual $8 - p$ form, type IIA contains all odd forms with $p = 1, \dots, 7$. We can also introduce a 9-form potential, but imposing the standard field equations sets its field strength $F^{(10)} = dC^{(9)}$ to a constant and so a 9-form carries no propagating degrees of freedom. We can think of massive type IIA as obtained by choosing this integration constant to be non-zero and equal to the mass parameter m . In the resulting theory, m enters as a cosmological constant and deforms the gauge transformations corresponding to the NS-NS b -field such that the RR 1-form transforms with a Stückelberg shift symmetry. It does not admit a maximally symmetric vacuum, but its most symmetric solution is the D8 brane solution that features 9-dimensional Poincaré invariance [21].

The type II double field theory of [5, 6] is formulated in terms of a Majorana-Weyl spinor of the ‘T-duality group’ $O(10, 10)$. This spinor representation is isomorphic to the set of all even or odd forms, depending on the chirality of the spinor, and so for type IIA the theory contains already a 9-form potential. However, the duality relations

$$*\hat{F}^{(10)} = -\hat{F}^{(0)} \equiv 0 \quad (1.1)$$

imply that its field strength is zero, because there is no non-trivial $\hat{F}^{(0)}$ due to the absence of ‘ (-1) -form’ potentials, and therefore the 9-form is on-shell determined to be pure gauge. Formally, one may introduce a (-1) -form potential $C^{(-1)}$ and then set $m = F^{(0)} = dC^{(-1)}$, as has been done in [22], but so far it has been unclear how to find a mathematically satisfactory interpretation of such objects. In this note we will show that a non-trivial 0-form field strength (and thus a mass parameter) is naturally included in the type II double field theory by assuming that the RR 1-form depends linearly on the winding coordinates,

$$C^{(1)}(x, \tilde{x}) = C_i(x)dx^i + m\tilde{x}_1dx^1, \quad (1.2)$$

where C_i and all other fields depend only on the 10-dimensional coordinates. We will see that the second term in (1.2) effectively acts as a (-1) -form and that the double field theory reduces precisely to massive type IIA.

It should be stressed that the consistency of the ansatz (1.2) is non-trivial, in fact, even unexpected, for the double field theory formulation requires the constraint that

$$\partial^M \partial_M A = \eta^{MN} \partial_M \partial_N A = 0, \quad \partial^M A \partial_M B = 0, \quad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1.3)$$

for all fields and gauge parameters A, B . Here, M, N, \dots are fundamental $O(10, 10)$ indices, with invariant metric η_{MN} , and the derivative $\partial_M = (\tilde{\partial}^i, \partial_i)$ combines the partial momentum and winding derivatives. This constraint is necessary for gauge invariance of the action and closure of the gauge algebra. In its *weak* form, which requires $\partial^M \partial_M = 2\tilde{\partial}^i \partial_i$ to annihilate all fields and parameters, it is a direct consequence of the level matching condition of closed string theory, and it allows for field configurations such as (1.2) that depend locally both on x and \tilde{x} . The double field theory constructions completed so far, however, impose the stronger form that requires also all products of fields and parameters to be annihilated by $\partial^M \partial_M$, corresponding to the second equation in (1.3). In this form the constraint implies that locally all fields depend only on half of the coordinates, and so (1.2) violates the strong constraint. Remarkably, as we will show here, the gauge transformations can be reformulated on the RR fields so that the strong constraint can be relaxed. It cannot be relaxed to the weak constraint as formulated above, but it is sufficient for the ansatz (1.2) to be consistent. In particular, this formulation guarantees that in the action and gauge transformations the linear \tilde{x} dependence drops out, such that the resulting theory has a conventional 10-dimensional interpretation.

This paper is organized as follows. In sec. 2 we briefly review the type II double field theory of [5, 6], and we give a rewriting of the gauge transformations that requires only a weaker constraint in a sense to be made precise. In sec. 3 we evaluate the double field theory for (1.2) and show that it reduces to the democratic formulation of massive type IIA. We then discuss T-duality for massive type II by evaluating the double field theory for a coordinate dependence in which one of the space-time coordinates is interchanged with a winding coordinate, corresponding to a single T-duality inversion. In the massless case this maps type IIA to type IIB, in agreement with their equivalence upon compactification on a circle. In the massive case, however, it maps type IIA into a non-covariant ‘massive’ version of type IIB. In this formulation, type IIB has only manifest 9-dimensional covariance but is fully diffeomorphism invariant, with the 10th diffeomorphism being deformed by the mass parameter.

2 Type II double field theory

2.1 Massless theory

We start by reviewing the double field theory formulation of massless type II theories. The NS-NS fields are the space-time metric g_{ij} , the Kalb-Ramond 2-form b_{ij} and the dilaton ϕ . The metric and b -field are encoded in the generalized metric

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}, \quad (2.1)$$

which is an element of $SO(10, 10)$, satisfying the constraint $\mathcal{H}^T = \mathcal{H}$ and transforming covariantly under $O(10, 10)$ according to its index structure. Moreover, we introduce an $O(10, 10)$ invariant dilaton density by $e^{-2d} = \sqrt{g}e^{-2\phi}$, where $g = |\det g_{ij}|$. We can think of \mathcal{H} as the fundamental field and view (2.1) as a particular parametrization, but in [5] we argued that we should rather view the fundamental field as an hermitian element of the two-fold covering group

$\text{Spin}(10, 10)$. Denoting this field by \mathbb{S} , we require

$$\mathbb{S} = \mathbb{S}^\dagger, \quad \mathbb{S} \in \text{Spin}(10, 10), \quad (2.2)$$

and *define* the generalized metric via the group homomorphism $\rho : \text{Spin}(10, 10) \rightarrow SO(10, 10)$ as $\mathcal{H} = \rho(\mathbb{S})$. With (2.2) we infer $\mathcal{H}^T = \rho(\mathbb{S}^\dagger) = \rho(\mathbb{S}) = \mathcal{H}$, and thus, as required, \mathcal{H} is a symmetric group element which generally can be parametrized as in (2.1).

In the democratic formulation to be employed here, the RR fields consist of differential forms of all odd (even) degrees for type IIA (IIB) and can be encoded in a Majorana-Weyl spinor χ of $O(10, 10)$. In order to see this, let us fix our conventions for $\text{Spin}(10, 10)$. The Clifford algebra is given by

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} \mathbf{1}. \quad (2.3)$$

Due to the off-diagonal form of the $O(10, 10)$ metric in (1.3), this algebra implies that

$$\psi_i \equiv \frac{1}{\sqrt{2}} \Gamma_i, \quad \psi^i \equiv \frac{1}{\sqrt{2}} \Gamma^i, \quad (2.4)$$

with $(\psi_i)^\dagger = \psi^i$, satisfy the canonical anti-commutation relations of fermionic raising and lowering operators ψ^i and ψ_i , respectively,

$$\{\psi_i, \psi^j\} = \delta_i^j, \quad \{\psi_i, \psi_j\} = 0, \quad \{\psi^i, \psi^j\} = 0. \quad (2.5)$$

Thus, the spinor representation can be constructed by introducing a Clifford vacuum $|0\rangle$, satisfying $\psi_i|0\rangle = 0$ for all i , and acting with the raising operators ψ^i . A spinor is then given by a general state

$$\chi = \sum_{p=0}^{10} \frac{1}{p!} C_{i_1 \dots i_p} \psi^{i_1} \dots \psi^{i_p} |0\rangle, \quad (2.6)$$

with coefficients that are fully antisymmetric tensors and that will be identified with the RR p -forms $C^{(p)}$. A spinor satisfying a chirality condition consists either of only odd or only even forms, and in the following we assume that χ has a fixed chirality. The Dirac operator

$$\not{D} \equiv \frac{1}{\sqrt{2}} \Gamma^M \partial_M = \psi^i \partial_i + \psi_i \tilde{\partial}^i, \quad (2.7)$$

acts naturally on spinors and can be viewed as the $O(10, 10)$ invariant extension of the exterior derivative. First, for $\tilde{\partial} = 0$ it only acts via differentiating with respect to x^i and increasing the form degree by one. Second, it squares to zero,

$$\not{D}^2 = \frac{1}{2} \Gamma^M \Gamma^N \partial_M \partial_N = \frac{1}{2} \eta^{MN} \partial_M \partial_N = 0, \quad (2.8)$$

using (2.3) and the constraint (1.3).

The type II double field theory, whose independent fields are \mathbb{S} , χ and d , is defined by the action

$$S = \int d^{10}x d^{10}\tilde{x} \left(e^{-2d} \mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\not{D}\chi)^\dagger \mathbb{S} \not{D}\chi \right), \quad (2.9)$$

supplemented by the self-duality constraint

$$\not{D}\chi = -\mathcal{K} \not{D}\chi, \quad \mathcal{K} \equiv C^{-1} \mathbb{S}, \quad (2.10)$$

where C denotess the charge conjugation matrix of $\text{Spin}(10,10)$. For the definition of the $O(10,10)$ invariant scalar $\mathcal{R}(\mathcal{H}, d)$ see eq. (4.24) in [3].

Let us next review the symmetries of this theory. It has a global T-duality invariance under $\text{Spin}^+(10,10)$, which is the component of the group connected to the identity. It is gauge invariant under

$$\delta_\lambda \chi = \not\partial \lambda, \quad (2.11)$$

where λ is a spinor of $\text{Spin}(10,10)$ with a fixed chirality opposite to the one of χ , such that odd (even) p -forms transform with even (odd) $(p-1)$ -form parameters. This symmetry leaves (2.9) and (2.10) manifestly invariant because of $\not\partial^2 = 0$. The double field theory is also invariant under a ‘generalized diffeomorphism’ symmetry spanned by a parameter $\xi^M = (\tilde{\xi}_i, \xi^i)$ that combines the usual diffeomorphism parameter ξ^i and the b -field gauge parameter $\tilde{\xi}_i$ into a fundamental $O(10,10)$ vector. This gauge symmetry acts on the fundamental fields as

$$\begin{aligned} \delta_\xi(e^{-2d}) &= \partial_M(\xi^M e^{-2d}), \\ \delta_\xi \mathcal{K} &= \xi^M \partial_M \mathcal{K} + \frac{1}{2} [\Gamma^{MN}, \mathcal{K}] \partial_M \xi_N, \\ \delta_\xi \chi &= \xi^M \partial_M \chi + \frac{1}{2} \partial_M \xi_N \Gamma^M \Gamma^N \chi, \end{aligned} \quad (2.12)$$

where $\Gamma^{MN} = \frac{1}{2}[\Gamma^M, \Gamma^N]$, and we have written the gauge variation of \mathbb{S} in terms of \mathcal{K} defined in (2.10). The gauge invariance under ξ^M transformations is non-trivial and requires the strong constraint (1.3). We stress that, in contrast, the identity (2.8) and thus the invariance under λ transformations requires only the weak constraint, i.e., that $\partial^M \partial_M$ annihilates χ and its gauge parameter, but not necessarily their products, which will be used below.

Next we discuss the double field theory evaluated for fields depending only on x^i , i.e., setting the winding derivatives to zero, $\tilde{\partial}^i = 0$. In order to compare the resulting theory with the conventional formulation, we need to choose a parametrization of \mathbb{S} in terms of g and b , as for \mathcal{H} in (2.1). We do so by decomposing (2.1)

$$\mathcal{H} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \equiv h_b^T h_{g^{-1}} h_b, \quad (2.13)$$

and then introducing spin representatives for each of the $SO(10,10)$ elements on the right-hand side. We then set

$$\mathbb{S} = S_{\mathcal{H}} \equiv S_b^\dagger S_g^{-1} S_b = e^{\frac{1}{2} b_{ij} \psi_i \psi_j} S_g^{-1} e^{-\frac{1}{2} b_{ij} \psi^i \psi^j}, \quad (2.14)$$

where we introduced the spin representative S_b of h_b in terms of fermionic oscillators [5]. The explicit form of the spin representative S_g of h_g can be found in [5]. By construction, $\mathbb{S} = S_{\mathcal{H}}$ projects under the group homomorphism ρ to \mathcal{H} in $SO(10,10)$, and it has been shown in [2,3] that the first term in (2.9) reduces to the standard low-energy action for the NS-NS sector.

We turn now to the RR sector. For $\tilde{\partial}^i = 0$ the Dirac operator $\not\partial$ acts like the exterior derivative and so $\not\partial \chi$ reduces to the conventional field strengths of the RR potentials in (2.6),

$$\not\partial \chi \Big|_{\tilde{\partial}=0} = \sum_p \frac{1}{p!} \partial_j C_{i_1 \dots i_p} \psi^j \psi^{i_1} \dots \psi^{i_p} |0\rangle = \sum_p \frac{1}{(p+1)!} F_{i_1 \dots i_{p+1}} \psi^{i_1} \dots \psi^{i_{p+1}} |0\rangle, \quad (2.15)$$

where we introduced in the last equation the components of the $(p+1)$ -form field strength,

$$F^{(p+1)} = dC^{(p)}. \quad (2.16)$$

In the action (2.9) the exponentials of b in (2.14) lead to the modified field strength [5, 6]

$$\widehat{F} = e^{-b^{(2)}} \wedge F = e^{-b^{(2)}} \wedge dC, \quad (2.17)$$

where we use a notation that combines all p -forms into a formal sum. Without repeating the derivation, we recall that the S_g factor in (2.14) effectively arranges the proper contractions with the space-time metric g in the action, so that the RR part of (2.9) reads [5, 6]

$$S_{\text{RR}} = -\frac{1}{4} \int d^{10}x \sqrt{g} \sum_p \frac{1}{p!} g^{i_1 j_1} \dots g^{i_p j_p} \widehat{F}_{i_1 \dots i_p} \widehat{F}_{j_1 \dots j_p} = \frac{1}{4} \sum_p \int \widehat{F}^{(p)} \wedge * \widehat{F}^{(p)}. \quad (2.18)$$

Similarly, the self-duality constraint (2.10) reduces to the conventional duality relations

$$\widehat{F}^{(p)} = -(-1)^{\frac{1}{2}p(p+1)} * \widehat{F}^{(10-p)}. \quad (2.19)$$

Thus, for the RR sector the double field theory action reduces to the sum of kinetic terms for all p -forms, which is supplemented by duality relations. This is the ‘democratic formulation’ originally introduced in [23], and which is on-shell equivalent to the conventional formulation of type IIA for odd forms or type IIB for even forms.

Let us finally review the gauge transformations for $\tilde{\partial}^i = 0$. The gauge variations (2.11), with parameter

$$\lambda = \sum_p \frac{1}{p!} \lambda_{i_1 \dots i_p} \psi^{i_1} \dots \psi^{i_p} |0\rangle, \quad (2.20)$$

reduce to

$$\delta_\lambda \chi = \psi^j \partial_j \lambda = \sum_p \frac{1}{(p-1)!} \partial_{[i_1} \lambda_{i_2 \dots i_p]} \psi^{i_1} \dots \psi^{i_p} |0\rangle, \quad (2.21)$$

which amounts to the standard abelian gauge symmetry of the RR p -forms,

$$\delta_\lambda C = d\lambda. \quad (2.22)$$

The gauge symmetry (2.12) parameterized by $\xi^M = (\tilde{\xi}_i, \xi^i)$ gives for ξ^i

$$\delta_\xi \chi = \left(\xi^j \partial_j + \partial_j \xi^k \psi^j \psi_k \right) \sum_p \frac{1}{p!} C_{i_1 \dots i_p} \psi^{i_1} \dots \psi^{i_p} |0\rangle, \quad (2.23)$$

which by use of the oscillator algebra (2.5) implies

$$\delta_\xi C_{i_1 \dots i_p} = \xi^j \partial_j C_{i_1 \dots i_p} + p \partial_{[i_1} \xi^j C_{j | i_2 \dots i_p]} \equiv \mathcal{L}_\xi C_{i_1 \dots i_p}. \quad (2.24)$$

This is the conventional diffeomorphism symmetry, acting infinitesimally via the Lie derivative. From (2.12) it also follows that the $C^{(p)}$ transform non-trivially under the b -field gauge parameter $\tilde{\xi}_i$,

$$\delta_{\tilde{\xi}} \chi = \partial_k \tilde{\xi}_l \psi^k \psi^l \chi = \sum_p \frac{1}{p!} \partial_{[i_1} \tilde{\xi}_{i_2} C_{i_3 \dots i_{p+2}]} \psi^{i_1} \dots \psi^{i_{p+2}} |0\rangle, \quad (2.25)$$

which implies

$$\delta_{\tilde{\xi}} C = d\tilde{\xi} \wedge C. \quad (2.26)$$

This means that the $C^{(p)}$ are redefinitions by the b -field of the more conventional RR fields, which are invariant under $\tilde{\xi}_i$. These are exactly the expected gauge symmetries for massless type II theories.

2.2 Reformulation of gauge symmetries

The gauge invariance of the type II double field theory requires for ξ^M transformations the strong form of the constraint (1.3), but for λ transformations only the weak constraint. Here, we will perform a change of basis for the gauge parameters such that, for the RR sector, the ξ^M transformations are consistent with a weaker form of the constraint.

We start by rewriting the ξ^M gauge transformation of χ as follows

$$\begin{aligned}\delta_\xi \chi &= \xi^M \partial_M \chi + \frac{1}{2} \partial_M \xi_N \Gamma^M \Gamma^N \chi \\ &= \xi^M \partial_M \chi - \frac{1}{2} \Gamma^M \Gamma^N \xi_N \partial_M \chi + \frac{1}{2} \Gamma^M \partial_M (\xi_N \Gamma^N \chi) .\end{aligned}\tag{2.27}$$

The last term is of the form of a field-dependent λ gauge transformation $\not\partial \lambda$ and can therefore be ignored. We then use the Clifford algebra in the second term,

$$\delta_\xi \chi = \xi^M \partial_M \chi - \frac{1}{2} (2\eta^{MN} - \Gamma^N \Gamma^M) \xi_N \partial_M \chi = \frac{1}{2} \Gamma^N \Gamma^M \xi_N \partial_M \chi .\tag{2.28}$$

Using the ‘slash’ notation (2.7),

$$\not\partial = \frac{1}{\sqrt{2}} \Gamma^M \partial_M , \quad \not\xi = \frac{1}{\sqrt{2}} \Gamma^M \xi_M ,\tag{2.29}$$

we finally get

$$\delta_\xi \chi = \not\xi \not\partial \chi ,\tag{2.30}$$

which is the form of the ξ^M gauge transformations we will use from now on.

We will show that, starting from (2.30), gauge invariance of the RR action and closure of the gauge algebra uses only the constraint

$$\eta^{MN} \partial_M \partial_N A = \partial^M \partial_M A = 0 , \quad A = \{ \chi, \lambda, \xi^M \} .\tag{2.31}$$

In particular, for this computation we do not need to use $\partial^M A \partial_M B = 0$. This observation does not imply, however, that the RR sector is ‘weakly constrained’ in the sense that fields but not their products need to satisfy the constraint. In fact, (2.30) is not a consistent transformation rule assuming that χ and ξ are weakly constrained. Before discussing this in more detail, we investigate some consequences of the form (2.30) of the gauge transformations.

The original gauge transformations have the property that a gauge parameter of the form $\xi^M = \partial^M \Theta$ is ‘trivial’ in that it generates no gauge transformation. After the above redefinition, this statement is modified. We compute

$$\delta_{\partial \Theta} \chi = \frac{1}{2} \Gamma^N \Gamma^M \partial_N \Theta \partial_M \chi = \frac{1}{2} \Gamma^N \partial_N (\Theta \Gamma^M \partial_M \chi) ,\tag{2.32}$$

assuming only the weaker form (2.31) of the constraint. Thus, the gauge variation (2.32) takes the form of a field-dependent λ gauge transformation,

$$\delta_{\partial \Theta} \chi = \not\partial \lambda , \quad \lambda = \Theta \not\partial \chi .\tag{2.33}$$

Therefore, the statement that $\xi^M = \partial^M \Theta$ leads to a trivial gauge transformation leaving the fields invariant has to be relaxed to the statement that it leaves the fields invariant *up to a* λ

gauge transformation, but it has the advantage that in this weaker form only the constraint (2.31) is required.

We compute next the gauge variation of $\not\partial\chi$ under (2.30), which is needed in order to verify gauge invariance,

$$\begin{aligned}\delta_\xi(\not\partial\chi) &= \not\partial(\xi\chi) = \not\partial\xi\chi + \frac{1}{2\sqrt{2}}\Gamma^M\Gamma^N\Gamma^P\xi_N\partial_M\partial_P\chi \\ &= \not\partial\xi\chi + \frac{1}{2\sqrt{2}}(2\eta^{MN} - \Gamma^N\Gamma^M)\Gamma^P\xi_N\partial_M\partial_P\chi.\end{aligned}\tag{2.34}$$

The last term contains $\Gamma^M\Gamma^P = \eta^{MP}$ and therefore vanishes by (2.31), while the second term reduces to $\xi^M\partial_M\not\partial\chi$. In total we have

$$\delta_\xi(\not\partial\chi) = \xi^M\partial_M\not\partial\chi + \not\partial\xi\chi.\tag{2.35}$$

This result agrees with the variation under the original form of the gauge transformations determined in [5] (as it should be, because the modification is a λ gauge transformation that leaves $\not\partial\chi$ invariant), but in the original derivation the strong constraint was used. As the proof of gauge invariance of the action and the self-duality constraint given in [5] requires only the transformation rule (2.35), we conclude that this proof uses only the weaker constraint (2.31).

Let us verify that also closure of the gauge transformations on χ requires only this weaker constraint. First, for the modified form of the gauge transformations there is no non-vanishing commutator between λ and ξ gauge transformations because $\not\partial\chi$ is λ -invariant. Thus, it remains to verify closure of the ξ^M transformations, for which we find

$$[\delta_{\xi^1}, \delta_{\xi^2}]\chi = \delta_{\xi^{12}}\chi + \delta_\lambda\chi.\tag{2.36}$$

Here,

$$\xi_{12}^M = [\xi_2, \xi_1]_C^M = \xi_2^N\partial_N\xi_1^M - \frac{1}{2}\xi_{2N}\partial^M\xi_1^N - (1 \leftrightarrow 2),\tag{2.37}$$

which is given by the ‘C-bracket’ that characterizes the closure of ξ^M transformations on the NS-NS fields [1, 2], and

$$\lambda = -\frac{1}{2}(\xi_2\xi_1 - \xi_1\xi_2)\not\partial\chi.\tag{2.38}$$

The verification of (2.36) is a straightforward though somewhat tedious exercise in gamma matrix algebra, which we defer to the appendix. The computation makes repeated use of the constraint (1.3), but only in its relaxed form (2.31). Thus, on the RR field χ all gauge symmetries close using only this weaker constraint.

We close this section by computing the form of these redefined ξ^M gauge transformations (2.30) for $\tilde{\partial}^i = 0$. For the diffeomorphism parameter ξ^i we find

$$\delta_\xi\chi = \xi^i\psi_i\psi^j\partial_j\chi = \sum_p \frac{1}{p!}\xi^i\partial_j C_{i_1\dots i_p}\psi_i\psi^j\psi^{i_1}\dots\psi^{i_p}|0\rangle.\tag{2.39}$$

Using the oscillator algebra (2.5) to simplify this, we obtain

$$\delta_\xi C_{i_1\dots i_p} = (p+1)\xi^j\partial_{[j}C_{i_1\dots i_p]} = \xi^j F_{ji_1\dots i_p}.\tag{2.40}$$

For the b -field gauge parameter $\tilde{\xi}_i$ one obtains

$$\delta_{\tilde{\xi}}\chi = \tilde{\xi}_i\psi^i\psi^j\partial_j\chi = \sum_p \frac{1}{p!}\tilde{\xi}_i\partial_j C_{i_1\dots i_p}\psi^i\psi^j\psi^{i_1}\dots\psi^{i_p}|0\rangle, \quad (2.41)$$

from which we read off

$$\delta_{\tilde{\xi}}C = \tilde{\xi} \wedge F. \quad (2.42)$$

The diffeomorphism symmetry in the form (2.40) is sometimes referred to as ‘improved diffeomorphisms’. They can be introduced for any p -form gauge field by adding to the familiar diffeomorphism symmetry (2.24) a field-dependent gauge transformation with $(p-1)$ -form parameter

$$\lambda_{i_1\dots i_{p-1}} = -\xi^j C_{ji_1\dots i_{p-1}}. \quad (2.43)$$

Similarly, (2.42) is obtained from the original $\tilde{\xi}$ transformation (2.26) by adding an abelian gauge transformation with parameter $\lambda = -\tilde{\xi} \wedge C$. Thus, the redefinition of the gauge transformations leading to (2.30) is precisely the double field theory analogue of the improved diffeomorphisms in conventional gauge theories. In this form the gauge field appears only under a derivative, which will be instrumental for the generalization we discuss next.

3 Massive type II theories

3.1 Massive type IIA

In the previous section we have seen that the proof of gauge invariance and closure of the gauge algebra uses only the weaker constraint (2.31) for the RR sector. Naively, this would allow for field configurations like

$$\chi(x, \tilde{x}) = \chi_0(x) + \chi_1(\tilde{x}), \quad (3.1)$$

where $\chi_{0,1}$ are arbitrary functions of their arguments, and similarly for the gauge parameters. However, as mentioned above, there is a subtlety, because the gauge variations (2.30) are not consistent assuming only the weak constraint. In fact, $\delta_{\xi}\chi$ on the left-hand side should satisfy the constraint, but with χ and ξ being weakly constrained their product on the right-hand side in general does not satisfy the constraint. Rather, one should introduce a projector that restricts to the part satisfying the weak constraint [1], while our computation above did not keep track of these projectors. After the insertion of projectors, the gauge invariance of the action and closure of the gauge algebra does not follow from our computation (and is most likely not true). Moreover, the RR fields interact with the NS-NS sector that is still strongly constrained, and so it is presumably inconsistent to have a weakly constrained RR sector. Thus, a complete relaxation of the strong constraint must await a resolution of this problem for the NS-NS sector. However, if we only assume the function χ_1 in (3.1) to depend linearly on \tilde{x} , the resulting gauge variations and field equations are independent of \tilde{x} , and therefore the constraint is satisfied without insertion of projectors. (In particular, the energy-momentum tensor of the RR fields depends only on $\not\partial\chi$ [5] and is thereby independent of \tilde{x} .) An ansatz with linear \tilde{x} dependence is therefore consistent, and we will investigate its consequences in what follows.

We will show that the type II double field theory defined by (2.9) and (2.10) leads to massive type IIA if we assume that the RR spinor χ depends on the 10-dimensional space-time coordinates and, in its 1-form part, also linearly on a winding coordinate. We thus write

$$\chi(x, \tilde{x}) = \left(\sum_p \frac{1}{p!} C_{i_1 \dots i_p}(x) \psi^{i_1} \dots \psi^{i_p} + m \tilde{x}_1 \psi^1 \right) |0\rangle, \quad (3.2)$$

where we assume that χ is of negative chirality such that the sum extends only over odd p . Here we have singled out a particular (winding) coordinate direction, but we stress that this choice is immaterial for the final result: we could have chosen any linear combination of the \tilde{x}_i , which would merely amount to a rescaling of the mass parameter m . Let us also note that it would be consistent to allow for a linear \tilde{x} dependence in other p -form parts, both in χ and in its gauge parameter λ . We will comment on this more general case below.

Let us next evaluate the field strength $\not\partial\chi$ for (3.2). In contrast to (2.15), the term $\psi_i \tilde{\partial}^i$ in $\not\partial$ acts now non-trivially,

$$\begin{aligned} \not\partial\chi &= \sum_p \frac{1}{p!} \partial_j C_{i_1 \dots i_p} \psi^j \psi^{i_1} \dots \psi^{i_p} |0\rangle + \psi_j \tilde{\partial}^j (m \tilde{x}_1) \psi^1 |0\rangle \\ &= \sum_p \frac{1}{(p+1)!} (p+1) \partial_{[i_1} C_{i_2 \dots i_{p+1}]} \psi^{i_1} \dots \psi^{i_{p+1}} |0\rangle + m |0\rangle \\ &\equiv \sum_p \frac{1}{(p+1)!} (F_m)_{i_1 \dots i_{p+1}} \psi^{i_1} \dots \psi^{i_{p+1}} |0\rangle, \end{aligned} \quad (3.3)$$

where we used the oscillator algebra (2.5). We observe that the non-trivial action of $\psi_i \tilde{\partial}^i$ leads to a reduction of the form degree such that the ‘1-form potential’ precisely leads to a non-vanishing 0-form field strength or, in other words, that the \tilde{x} dependent part acts effectively like a ‘(−1)-form’. The m -deformed field strengths defined in the last line of (3.3) then read

$$F_m^{(0)} = m, \quad F_m^{(p+1)} = F^{(p+1)} = dC^{(p)} \quad \text{for } p \geq 0. \quad (3.4)$$

In the action the modified field strengths (2.17) enter, which are now deformed according to (3.4),

$$\hat{F}_m = e^{-b^{(2)}} \wedge (dC + m). \quad (3.5)$$

This reads explicitly

$$\begin{aligned} \hat{F}_m^{(0)} &= m \\ \hat{F}_m^{(2)} &= F^{(2)} - m b^{(2)} \\ \hat{F}_m^{(4)} &= F^{(4)} - b^{(2)} \wedge F^{(2)} + \frac{1}{2} m b^{(2)} \wedge b^{(2)}, \quad \text{etc.} \end{aligned} \quad (3.6)$$

These are precisely the m -deformed field strengths appearing in massive type IIA, see, e.g., [21].

We turn now to the gauge symmetries acting on (3.2), starting with the λ -transformations (2.11). In analogy to (3.2) it is natural to allow here also for a linear \tilde{x} dependence in the 0-form part of λ , but such a contribution will be annihilated by $\not\partial$ due to $\psi_i |0\rangle = 0$. We note, however, that a linear \tilde{x} dependence in the higher-form components of λ can lead to a *rigid*

shift of the RR forms, which is trivially a global symmetry since all RR potentials appear under a derivative. We conclude that the λ gauge transformations are unchanged compared to the massless case (2.22). The ξ^M transformation (2.30) evaluated for the diffeomorphism parameter ξ^i yields no new contribution since

$$\delta_{\xi^i} \chi \Big|_{\partial_i=0} = \xi^i \psi_i \psi_j \tilde{\partial}^j \chi = 0, \quad (3.7)$$

due to the action of two annihilation operators ψ_i on (3.2). Thus, the diffeomorphism symmetry is given by (2.40), as for $m = 0$. Finally, the gauge transformation of the b -field gauge parameter $\tilde{\xi}_i$ receives a non-trivial modification,

$$\delta_{\tilde{\xi}_i} \chi \Big|_{\partial_i=0} = \tilde{\xi}_i \psi^i \psi_j \tilde{\partial}^j \chi = m \tilde{\xi}_i \psi^i \psi_1 \psi^1 |0\rangle = m \tilde{\xi}_i \psi^i |0\rangle. \quad (3.8)$$

Together with the gauge transformation (2.42) for $m = 0$ we thus obtain

$$\delta_{\tilde{\xi}} C = \tilde{\xi} \wedge dC + m \tilde{\xi}. \quad (3.9)$$

Therefore, for $m \neq 0$ the RR 1-form $C^{(1)}$ transforms with a Stückelberg shift symmetry under the b -field gauge transformations, which is precisely the expected result for massive type IIA [21]. The modified field strengths \hat{F}_m are manifestly invariant under the λ gauge transformations. The invariance under $\tilde{\xi}$ transformations can be easily verified with $\delta_{\tilde{\xi}} b^{(2)} = d\tilde{\xi}$,

$$\begin{aligned} \delta_{\tilde{\xi}} \hat{F}_m &= \delta_{\tilde{\xi}} (e^{-b^{(2)}} \wedge (dC + m)) = -d\tilde{\xi} \wedge \hat{F}_m + e^{-b^{(2)}} \wedge d(\tilde{\xi} \wedge dC + m\tilde{\xi}) \\ &= -d\tilde{\xi} \wedge \hat{F}_m + e^{-b^{(2)}} \wedge d\tilde{\xi} \wedge (dC + m) = -d\tilde{\xi} \wedge \hat{F}_m + d\tilde{\xi} \wedge \hat{F}_m = 0. \end{aligned} \quad (3.10)$$

Let us now consider the double field theory action and duality relations (2.9) and (2.10), evaluated for (3.2), and compare with the dynamics of massive type IIA. As in (2.18), the action reduces to the sum of kinetic terms, but here for the modified field strengths (3.5),

$$\mathcal{L}_{\text{RR}} = \frac{1}{4} \sum_{p=0}^{10} \hat{F}_m^{(p)} \wedge * \hat{F}_m^{(p)} = \frac{1}{4} \sum_{p \geq 1} \hat{F}_m^{(p)} \wedge * \hat{F}_m^{(p)} + \frac{1}{4} m^2 * 1. \quad (3.11)$$

The action contains now also the 0-form field strength, which contributes a cosmological term proportional to m^2 , as made explicit in the second equation. Moreover, we can use the Stückelberg gauge symmetry (3.9) with parameter $\tilde{\xi}$ to set $C^{(1)} = 0$. From the second equation in (3.6) we then infer that the kinetic term for $C^{(1)}$ reduces to a mass term for the b -field. Thus, the b -field becomes massive by ‘eating’ the RR 1-form.

The self-duality constraint (2.10) reduces to the same duality relations as in (2.19), again with all field strengths being m -deformed,

$$\hat{F}_m^{(p)} = -(-1)^{\frac{1}{2}p(p+1)} * \hat{F}_m^{(10-p)}. \quad (3.12)$$

This democratic formulation is equivalent to the conventional formulation of massive type IIA. In the following we compare the two formulations in a little more detail.

The RR action of massive type IIA in the standard formulation is given by [17,21]

$$\begin{aligned}
S_{\text{RR}} = & \frac{1}{2} \int \left(\widehat{F}_m^{(2)} \wedge * \widehat{F}_m^{(2)} + \widehat{F}_m^{(4)} \wedge * \widehat{F}_m^{(4)} + m^2 * 1 \right) \\
& + \frac{1}{2} \int \left(b^{(2)} (dC^{(3)})^2 - (b^{(2)})^2 dC^{(1)} dC^{(3)} + \frac{1}{3} (b^{(2)})^3 (dC^{(1)})^2 \right. \\
& \left. + \frac{1}{3} m (b^{(2)})^3 dC^{(3)} - \frac{1}{4} m (b^{(2)})^4 dC^{(1)} + \frac{1}{20} m^2 (b^{(2)})^5 \right), \tag{3.13}
\end{aligned}$$

where for simplicity we have omitted all wedge products between forms in the topological Chern-Simons terms S_{CS} in the second and third line.¹ We note in passing that this Chern-Simons action simplifies significantly if we formally introduce a (-1) -form $C^{(-1)}$ and then define

$$\widehat{A} = e^{-b^{(2)}} \wedge (C + C^{(-1)}), \tag{3.14}$$

where C still represents the formal sum of all (odd) p -forms with $p \geq 1$. The Chern-Simons action can then simply be written as

$$S_{\text{CS}} = \frac{1}{2} \int b^{(2)} \wedge d\widehat{A}^{(3)} \wedge d\widehat{A}^{(3)}. \tag{3.15}$$

More precisely, expanding (3.15) according to (3.14), the resulting action can be written, up to total derivatives, such that $C^{(-1)}$ enters only under an exterior derivative, and then setting $m = dC^{(-1)}$ reproduces precisely the Chern-Simons terms in (3.13). Formally, this drastic simplification can be understood as a consequence of the b -field gauge transformations (3.9), which we rewrite here as

$$\delta_{\tilde{\xi}} C = \tilde{\xi} \wedge d(C + C^{(-1)}) = d\tilde{\xi} \wedge (C + C^{(-1)}) - d(\tilde{\xi} \wedge (C + C^{(-1)})). \tag{3.16}$$

The last term takes the form of a field-dependent λ gauge transformation and can thus be ignored. The \widehat{A} defined in (3.14) is then $\tilde{\xi}$ gauge invariant,

$$\delta_{\tilde{\xi}} \widehat{A} = -d\tilde{\xi} \wedge e^{-b^{(2)}} \wedge (C + C^{(-1)}) + e^{-b^{(2)}} \wedge (d\tilde{\xi} \wedge (C + C^{(-1)})) = 0, \tag{3.17}$$

where we have taken $C^{(-1)}$ to be gauge invariant. From this we infer that (3.15) is the only term invariant under $\tilde{\xi}$ gauge transformations (up to a boundary term). Note that we could have included the (-1) -form potential into the sum of all p -forms, in which case the gauge transformations would be formally as in the massless case.

We have verified the exact equivalence between the equations of motion following from (3.13) and those derived by varying (3.11) and then supplementing them by the duality relations (3.12). For the Einstein equations this is easy to see because the Chern-Simons terms that are present in the conventional formulation do not contribute to the variation of the metric. The energy-momentum tensor then agrees for both formulations owing to the relative factor of $\frac{1}{2}$ between the kinetic terms in (3.11) and (3.13), which compensates for the doubling of fields in

¹This action differs from eq. (2.8) of [21] in certain numerical factors, which is due to different conventions regarding differential forms. Moreover, there is a mismatch of a relative factor of $\frac{1}{2}$ between kinetic and Chern-Simons terms, but (3.13) is consistent with [17].

the democratic formulation. For the field equations of the p -forms the on-shell equivalence is a consequence of the Bianchi identities

$$d\hat{F}_m^{(p)} = -H^{(3)} \wedge \hat{F}_m^{(p-2)}, \quad (3.18)$$

following from (3.5). More precisely, the duality relations yield the second-order field equations as integrability conditions of $d^2 = 0$, including the required source terms originating from the Chern-Simons terms in the conventional formulation. Thus, the double field theory leads precisely to massive type IIA.

3.2 T-duality and massive type IIB

We discuss now the double field theory evaluated for fields depending on coordinates that result from the 10-dimensional space-time coordinates x^i by a T-duality inversion. The $O(10, 10)$ invariance of the constraint (1.3) implies that fields resulting by an $O(10, 10)$ transformation from fields depending only on the x^i (thereby satisfying the constraint) also satisfy the constraint. For instance, we may perform a single T-duality inversion in one direction, which exchanges a ‘momentum coordinate’ x^i with the corresponding ‘winding coordinate’ \tilde{x}_i . The double field theory evaluated for this field configuration then reduces to the T-dual theory. If it reduces to type IIA in one ‘T-duality frame’, it reduces to type IIB in the other frame, when expressed in the right T-dual field variables [5]. The mapping of (massless) type IIA into type IIB under T-duality can therefore be discussed without reference to dimensional reduction, while in the usual approach this relation is inferred from the equivalence of type IIA and type IIB upon reduction on a circle [24].

Our task is now to see how this generalizes in the massive case. The usual point of view is as follows [21]. Massive type IIA reduced on a circle leads to a massive $N = 2$ theory in nine dimensions, but there is no corresponding massive deformation of type IIB that could lead to the same nine-dimensional theory upon standard reduction. Rather, to identify the proper T-duality rules one has to perform a Scherk-Schwarz reduction [25] of massless type IIB, which introduces a mass parameter and leads to the same massive $N = 2$ theory in nine dimensions. In contrast, in double field theory the T-dual theory is identified without any dimensional reduction, as we discussed above, and so the puzzle arises what the T-dual to massive type IIA is if there is no massive type IIB in ten dimensions.

In order to address this issue let us analyze the double field theory evaluated for fields in which one space-time coordinate, say x^{10} , is replaced by the corresponding winding coordinate. We split the coordinates as $x^i = (x^\mu, x^{10})$, $\mu = 1, \dots, 9$, and replace (3.2) by the ansatz

$$\chi(x, \tilde{x}) = \left(\sum_p \frac{1}{p!} C_{i_1 \dots i_p}(x^\mu, \tilde{x}_{10}) \psi^{i_1} \dots \psi^{i_p} + m \tilde{x}_1 \psi^1 \right) |0\rangle, \quad (3.19)$$

where again the sum extends over all odd p . In the massless case the double field theory reduces to type IIB, which can be made manifest by performing a field redefinition that takes the form of a T-duality inversion in the 10th direction [5].² This T-duality transformation acts on the

²Here we assume that x^{10} is a space-like direction, $g_{10,10} > 0$. For T-dualities along time-like directions the dual theories are the so-called type II* theories [5, 6], which have a reversed sign for the RR kinetic terms [26]. Similarly, the double field theory discussed here contains also a massive type IIA*.

RR spinor via the spin representative $S_{10} = \psi^{10} + \psi_{10}$, i.e., we define

$$\chi' = S_{10} \chi = \left(\sum_p \frac{1}{p!} C'_{i_1 \dots i_p} \psi^{i_1} \dots \psi^{i_p} + m \tilde{x}_1 (\psi^{10} + \psi_{10}) \psi^1 \right) |0\rangle, \quad (3.20)$$

where in the first term we introduced redefined variables denoted by C' . As S_{10} is linear in the fermionic oscillators the sum extends now over all *even* p . Specifically, one finds (compare eq. (6.41) in [6])

$$C'_{i_1 \dots i_p} = \begin{cases} C_{\mu_2 \dots \mu_p} & \text{if } i_1 = 10, i_2 = \mu_2, \dots, i_p = \mu_p \\ C_{10\mu_1 \dots \mu_p} & \text{if } i_1 = \mu_1, \dots, i_p = \mu_p. \end{cases} \quad (3.21)$$

Thus, the dual field variables are obtained by adding or deleting the special index, thereby mapping odd forms into even forms, as required for the transition from type IIA to type IIB. By performing this field redefinition (and renaming the coordinates) one infers that evaluating the theory for fields depending on x^μ and \tilde{x}_{10} is equivalent to evaluating the theory for fields depending on x^i , but with the opposite chirality for the spinor, i.e., replacing odd forms by even forms. (See sec. 6.2 in [6] for more details.) Now, in the massive case we have to take into account the second term in (3.20), which reduces to $m \tilde{x}_1 \psi^{10} \psi^1$. Thus, our task is to evaluate the double field theory for

$$\chi(x, \tilde{x}) = \left(\sum_p \frac{1}{p!} C_{i_1 \dots i_p}(x) \psi^{i_1} \dots \psi^{i_p} + m \tilde{x}_1 \psi^{10} \psi^1 \right) |0\rangle, \quad (3.22)$$

dropping the primes from now on. In other words, we have to evaluate the double field theory for a field configuration in which the 2-form part depends now linearly on \tilde{x} ,

$$(\chi(x, \tilde{x})|_{2\text{-form}})_{ij} = C_{ij}(x) + 2m \tilde{x}_1 \delta_{[i}^{10} \delta_{j]}^1, \quad (3.23)$$

with all other fields still depending only on the 10-dimensional space-time coordinates.

We start by computing the field strength

$$F = \oint \chi = F_{m=0} - \psi_1 \tilde{\partial}^1 (m \tilde{x}_1) \psi^1 \psi^{10} |0\rangle = F_{m=0} - m \psi^{10} |0\rangle. \quad (3.24)$$

Therefore, the field strength of the RR 0-form $C^{(0)}$ gets modified in the 10th component,

$$F^{(1)} = dC^{(0)} - m dx^{10} \quad \Leftrightarrow \quad F_i = \partial_i C^{(0)} - m \delta_i^{10}, \quad (3.25)$$

while all other field strengths $F^{(p)}$, $p \neq 1$, remain unchanged. The ‘hatted’ field strength (2.17) then receives corresponding modifications,

$$\hat{F} = e^{-b^{(2)}} \wedge (dC - m dx^{10}), \quad (3.26)$$

and thus in components

$$\hat{F}^{(3)} = F^{(3)} - b^{(2)} \wedge dC^{(0)} + m b^{(2)} \wedge dx^{10}, \quad \text{etc.} \quad (3.27)$$

The dynamics is described by the same action (2.18) and duality relations (2.19) as before, but with all field strengths replaced by their m -deformed version (3.26).

This theory breaks manifest 10-dimensional covariance in that the 10th coordinate is treated on a different footing in (3.25). We observe, however, that this theory can be obtained from standard (covariant) type IIB by performing the redefinition

$$C^{(0)} \rightarrow C^{(0)} - mx^{10} , \quad (3.28)$$

as is apparent from (3.25). Thus, the ‘deformation’ induced by the m -dependent 2-form contribution in (3.22) can be absorbed into a redefinition of the lower RR form $C^{(0)}$, and therefore the obtained theory is nothing but standard type IIB after a somewhat peculiar (non-covariant) redefinition. For this reason we do not introduce a new symbol for the ‘deformed’ field strengths.

In order to understand the consequences of the non-covariance let us inspect the gauge symmetries. As above, the λ gauge transformations are unchanged compared to the massless case. The gauge transformations (2.30) parametrized by ξ^M applied to (3.22) give

$$\begin{aligned} \delta_\xi \chi &= (\psi^i \tilde{\xi}_i + \psi_i \xi^i) \not{\partial} \chi = \delta_\xi \chi \Big|_{m=0} - m(\tilde{\xi}_i \psi^i \psi^{10} + \xi^i \psi_i \psi^{10}) |0\rangle \\ &= \delta_\xi \chi \Big|_{m=0} - m(\tilde{\xi}_\mu \psi^\mu \psi^{10} + \xi^{10}) |0\rangle . \end{aligned} \quad (3.29)$$

We read off the m -deformed gauge transformations which are modified on $C_{\mu 10}$,

$$\delta_{\tilde{\xi}} C_{\mu 10} = 2\tilde{\xi}_{[\mu} F_{10], m=0} - m\tilde{\xi}_\mu = 2\tilde{\xi}_{[\mu} F_{10]} , \quad (3.30)$$

and on $C^{(0)}$

$$\delta_\xi C^{(0)} = \xi^j \partial_j C^{(0)} - m\xi^{10} = \xi^j F_j , \quad (3.31)$$

where we used (3.25) for both equations in the last step. Thus, the nine-component parameter $\tilde{\xi}_\mu$ acts as a Stückelberg symmetry on the off-diagonal RR 2-form components, while the 10th diffeomorphism parameter ξ^{10} acts as a Stückelberg symmetry on the RR 0-form. The field strength of $C_{\mu 10}$ read off from (3.27),

$$\hat{F}_{\mu\nu 10} = 2\partial_{[\mu} C_{\nu] 10} + mb_{\mu\nu} + \partial_{10} C_{\mu\nu} - b_{\mu\nu} \partial_{10} C^{(0)} - 2b_{10[\mu} \partial_{\nu]} C^{(0)} , \quad (3.32)$$

is invariant under the $\tilde{\xi}_\mu$ shift symmetry. Moreover, (3.25) is invariant under ξ^{10} , i.e., the theory is diffeomorphism invariant under $x^{10} \rightarrow x^{10} - \xi^{10}(x)$ and (3.31),

$$\delta_{\xi^{10}} F^{(1)} = -m d\xi^{10} + m d\xi^{10} = 0 . \quad (3.33)$$

Thus, despite the non-covariant formulation that treats the 10th direction on a different footing, the theory is still fully diffeomorphism invariant, as it should be in view of the fact that it results from standard type IIB by the redefinition (3.28). Since this invariance under non-covariant diffeomorphisms is somewhat unconventional, let us also verify this for the component form given in (3.25),

$$\begin{aligned} \delta_\xi F_i &= \partial_i (\xi^j \partial_j C^{(0)} - m\xi^{10}) = \xi^j \partial_j (\partial_i C^{(0)}) + \partial_i \xi^j \partial_j C^{(0)} - m\partial_i \xi^{10} \\ &= \xi^j \partial_j F_i + \partial_i \xi^j (\partial_j C^{(0)} - m\delta_j^{10}) = \xi^j \partial_j F_i + \partial_i \xi^j F_j . \end{aligned} \quad (3.34)$$

Thus, the m -deformed field strength transforms under the m -deformed diffeomorphisms (3.31) with the usual Lie derivative of a 1-form field strength. Therefore, the action and duality relations build with this field strength are diffeomorphism invariant.

To summarize, we have identified the 10-dimensional theory that is the T-dual to massive type IIA and that can be seen as a ‘massive’ formulation of type IIB. It is unconventional in that the 10-dimensional diffeomorphism symmetry is not realized in the usual way, but non-linearly in the 10th direction. This is, however, analogous to the deformation of the gauge transformation of $C^{(1)}$ under the b -field gauge parameter in massive type IIA, and since the diffeomorphisms and b -field gauge symmetries are on the same footing in double field theory this result is not surprising.

Let us now discuss the physical content. We can choose a gauge for the $\tilde{\xi}_\mu$ Stückelberg symmetries by setting $C_{\mu 10} = 0$. From (3.32) we then infer that their kinetic terms give mass terms for the 9-dimensional components of the b -field, rendering these components massive. This is analogous to massive type IIA, but in the latter case the full 10-dimensional b -field becomes massive, carrying 36 massive degrees of freedom, while here only the 9-dimensional components become massive, carrying 28 massive degrees of freedom. It turns out that the 8 missing degrees of freedom are carried instead by the Kaluza-Klein vector field. In order to see this, let us perform a Kaluza-Klein decomposition of the kinetic term involving $C^{(0)}$ (but we stress that we are not performing a reduction in that the fields still depend on all 10 coordinates). The standard Kaluza-Klein decomposition of the (inverse) metric reads

$$g^{ij} = \begin{pmatrix} \gamma^{\mu\nu} & -A^\mu \\ -A^\nu & \ell^{-1} + A^\rho A_\rho \end{pmatrix}, \quad (3.35)$$

where $\gamma_{\mu\nu}$ denotes the 9-dimensional metric, A_μ is the Kaluza-Klein vector and ℓ the Kaluza-Klein scalar. If we choose a gauge for the ξ^{10} Stückelberg symmetry by setting $C^{(0)} = 0$, we infer with (3.25) that the relevant term in the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}g^{ij}F_iF_j = -\frac{1}{4}\sqrt{g}g^{10,10}F_{10}F_{10} = -\frac{1}{4}m^2\sqrt{\gamma}\sqrt{\ell}(\ell^{-1} + A^\mu A_\mu). \quad (3.36)$$

Therefore, the Kaluza-Klein vector receives a mass term and so becomes massive by ‘eating’ the RR scalar $C^{(0)}$, thus carrying 8 massive degrees of freedom.

We have to point out that the above analysis of the physical content was somewhat naive. In fact, one may wonder why this theory, if obtained from massless type IIB by the mere redefinition (3.28), exhibits a spectrum that is rather different from the usual physical content of type IIB, e.g., with (parts of) the b -field becoming massive and a cosmological term in (3.36). The point is that such a classification of the masses of various fields is only meaningful with respect to a particular background. Type IIB admits a 10-dimensional Minkowski solution, with all field strengths zero in the background, and it is with respect to this background that the b -field is massless. Now, after the redefinition (3.28) the theory of course still admits the same Minkowski vacuum, but now we have to switch on a ‘background flux’ in order to realize this solution,

$$\langle g_{ij} \rangle = \eta_{ij}, \quad \langle dC^{(0)} \rangle = m dx^{10}, \quad (3.37)$$

because only then we have $\langle \hat{F} \rangle = 0$ in the Einstein equations, as follows with (3.25). Around this background, the b -field is still massless.

Thus, there is no conflict of our above analysis of ‘massive’ type IIB with the usual way type IIB is presented. The presence of massive fields just means that the background space-time we

consider is *not* flat space, but rather a background that is appropriate for the comparison to the T-dual massive type IIA. In fact, massive type IIA does not admit a Minkowski (or AdS) vacuum, but instead the D8-brane solution that is invariant under the 9-dimensional Poincaré group corresponding to its world-volume [21]. The T-dual configuration is the D7-brane solution of type IIB, which is only invariant under the 8-dimensional Poincaré group [21], and the above analysis has to be understood with respect to such a background.

Let us close this section by comparing our result with the usual story that relates massive type IIA to the Scherk-Schwarz reduction of massless type IIB [21, 22]. In Scherk-Schwarz reduction one allows some fields to depend non-trivially on the internal coordinates in such a way that this dependence drops out in the effective lower-dimensional theory. For the Scherk-Schwarz reduction of type IIB to nine dimensions relevant for T-duality, the Kaluza-Klein ansatz allows for a linear x^{10} dependence for the RR scalar $C^{(0)}$,

$$C^{(0)}(x^\mu, x^{10}) = c^{(0)}(x^\mu) - mx^{10}, \quad (3.38)$$

where $c^{(0)}$ denotes the nine-dimensional field. For all other fields the ansatz is as for circle reductions, i.e., the fields are simply assumed to be independent of x^{10} . In the resulting action the dependence on x^{10} drops out, leaving a massive deformation of the usual circle reduction of type IIB.

Instead of this Scherk-Schwarz reduction one may first perform the redefinition (3.28) and then employ a standard reduction, as is apparent by comparing (3.38) with (3.28). We conclude that the Scherk-Schwarz reduction of massless type IIB gives the same 9-dimensional theory as the conventional reduction of the ‘massive’ formulation of type IIB. Thus, our results are consistent with [21, 22], and the formulation of type IIB that appears naturally in double field theory is already adapted to the Scherk-Schwarz reduction.³

4 Concluding remarks

In this paper we have shown that the type II double field theory defined by (2.9) and (2.10) can be extended by slightly relaxing the constraint (1.3) such that the RR fields may depend simultaneously on all 10-dimensional space-time coordinates and linearly on the winding coordinates. In case that only the RR 1-form carries such a dependence, the double field theory reduces precisely to the massive type IIA theory. We have shown that the T-dual configuration corresponds to the case that the RR 2-form (3.23) of type IIB carries such a dependence. This gives rise to a ‘massive’ version of type IIB, whose circle reduction to nine dimensions yields the same theory as the Scherk-Schwarz reduction of conventional type IIB. This massive formulation of type IIB is still invariant under 10-dimensional diffeomorphisms, with the 10th diffeomorphism being deformed by the mass parameter.

Here we have only considered a non-trivial \tilde{x} dependence for the RR 1-form of type IIA and, in the T-dual situation, for the RR 2-form of type IIB. It is also consistent with the relaxed constraint to have a linear \tilde{x} dependence for all higher RR forms $C^{(p)}$. Such an ansatz would lead to a multiple parameter family of ‘massive’ and non-covariant type II theories, but as for

³We thank Eric Bergshoeff for discussions on this point.

the type IIB case discussed above all of these deformations can be absorbed into a redefinition of the lower RR form $C^{(p-2)}$, as in (3.28). The only exception is the RR 1-form which is distinguished because it leads to a *covariant* massive deformation in that it merely deforms the 0-form field strength by the scalar mass parameter. This deformation cannot be absorbed into a redefinition, precisely because there is no ‘ (-1) -form’ potential available that could be redefined. This is in agreement with the fact that no massive deformations of maximal 10-dimensional supergravity are known besides massive type IIA. Even though (-1) -forms do not exist in the usual framework of differential geometry (such that they appear in conventional type II theories at most in a very formal sense), the type II double field theory suggests a natural and concrete interpretation: (-1) -forms are 1-forms depending on the dual coordinates.

Finally, let us stress that so far we have only established the existence of the bosonic sector of the massive type II theories since the supersymmetric extension of type II double field theory has not yet been constructed, but we strongly suspect that it exists. In fact, the approach of Siegel, which is equivalent to the NS-NS sector of double field theory, is already formulated in superspace, thereby establishing $N = 1$ supersymmetry [7]. Moreover, the recent work [12] presents a rewriting of the $N = 2$ fermionic terms and supersymmetry variations in the context of generalized geometry. While this does not yet prove the existence of an $N = 2$ supersymmetric extension of double field theory, since the coordinates are not doubled in generalized geometry, it provides strong evidence. These matters are currently under investigation.

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A Proof of the gauge algebra

Here we prove that the gauge transformations (2.30) close according to (2.36), using only the weaker constraint (2.31). We compute

$$\begin{aligned} [\delta_{\xi^1}, \delta_{\xi^2}] \chi &= \delta_{\xi^1} \left(\frac{1}{2} \Gamma^N \Gamma^M \xi_{2N} \partial_M \chi \right) - (1 \leftrightarrow 2) \\ &= \frac{1}{4} \Gamma^N \Gamma^M \Gamma^P \Gamma^Q \xi_{2N} \partial_M (\xi_{1P} \partial_Q \chi) - (1 \leftrightarrow 2) . \end{aligned} \tag{A.1}$$

Let us work out structures with $\partial \chi$ and $\partial^2 \chi$ separately. Consider

$$\begin{aligned} \frac{1}{8} \Gamma^N \Gamma^M \Gamma^P \Gamma^Q \xi_{2N} \xi_{1P} \partial_M \partial_Q \chi &= \frac{1}{8} (\Gamma^P \Gamma^Q \Gamma^N \Gamma^M + [\Gamma^N \Gamma^M, \Gamma^P \Gamma^Q]) \xi_{2N} \xi_{1P} \partial_M \partial_Q \chi \\ &= \frac{1}{8} (\Gamma^P \Gamma^Q \Gamma^N \Gamma^M - 2(\eta^{MP} \Gamma^{QN} + \eta^{NQ} \Gamma^{PM})) \xi_{2N} \xi_{1P} \partial_M \partial_Q \chi , \end{aligned} \tag{A.2}$$

where we used the constraint (2.31) for χ and the symmetry in M, Q . If we antisymmetrize in $1 \leftrightarrow 2$ and use the symmetry in M, Q , we infer that the term on the left-hand side in the first line is minus the first term in the second line. Similarly, upon antisymmetrization $1 \leftrightarrow 2$, the final two terms in the second line are equal. Rearranging terms, we thus get

$$\begin{aligned} \frac{1}{4}\Gamma^N\Gamma^M\Gamma^P\Gamma^Q\xi_{2N}\xi_{1P}\partial_M\partial_Q\chi - (1 \leftrightarrow 2) &= \frac{1}{2}\Gamma^M\Gamma^P\xi_2^N\xi_{1P}\partial_M\partial_N\chi - (1 \leftrightarrow 2) \\ &= \frac{1}{2}\Gamma^M\partial_M(\xi_2^N\Gamma^P\xi_{1P}\partial_N\chi) - \frac{1}{2}\Gamma^M\Gamma^P\partial_M(\xi_2^N\xi_{1P})\partial_N\chi - (1 \leftrightarrow 2), \end{aligned} \quad (\text{A.3})$$

where we used in the first equality that a term with η^{MP} is zero by the antisymmetry $1 \leftrightarrow 2$. The first term in the second line takes the form of a field-dependent λ gauge transformation.

We turn now to the terms proportional to $\partial\chi$ in (A.1),

$$\frac{1}{4}\Gamma^N\Gamma^M\Gamma^P\Gamma^Q\xi_{2N}\partial_M\xi_{1P}\partial_Q\chi = \frac{1}{2}\Gamma^P\Gamma^Q\xi_2^M\partial_M\xi_{1P}\partial_Q\chi - \frac{1}{4}\Gamma^M\Gamma^N\Gamma^P\Gamma^Q\xi_{2N}\partial_M\xi_{1P}\partial_Q\chi, \quad (\text{A.4})$$

where we used the Clifford algebra for $\Gamma^N\Gamma^M$. The first term on the right-hand side takes the form of a ξ^M gauge transformation (2.30). The second term can be re-written as

$$\begin{aligned} -\frac{1}{4}\Gamma^M\Gamma^N\Gamma^P\Gamma^Q\xi_{2N}\partial_M\xi_{1P}\partial_Q\chi &= -\frac{1}{4}\Gamma^M(\Gamma^Q\Gamma^N\Gamma^P + [\Gamma^N\Gamma^P, \Gamma^Q])\xi_{2N}\partial_M\xi_{1P}\partial_Q\chi \\ &= -\frac{1}{4}\Gamma^M\Gamma^Q\Gamma^N\Gamma^P\xi_{2N}\partial_M\xi_{1P}\partial_Q\chi + \frac{1}{2}\Gamma^M(\eta^{QN}\Gamma^P - \eta^{QP}\Gamma^N)\xi_{2N}\partial_M\xi_{1P}\partial_Q\chi. \end{aligned} \quad (\text{A.5})$$

Antisymmetrizing $1 \leftrightarrow 2$, the last term in the second line gives

$$\begin{aligned} \frac{1}{2}\Gamma^M\Gamma^P\xi_2^N\partial_M\xi_{1P}\partial_N\chi - \frac{1}{2}\Gamma^M\Gamma^N\xi_{2N}\partial_M\xi_1^P\partial_P\chi - (1 \leftrightarrow 2) \\ = \frac{1}{2}\Gamma^M\Gamma^P\partial_M(\xi_2^N\xi_{1P})\partial_N\chi - (1 \leftrightarrow 2), \end{aligned} \quad (\text{A.6})$$

which cancels against the same structure in (A.3). The first term in the second line of (A.5), antisymmetrized in $1 \leftrightarrow 2$, can be simplified as follows

$$\begin{aligned} -\frac{1}{4}\Gamma^M\Gamma^Q(\eta^{NP} + \Gamma^{NP})(\xi_{2N}\partial_M\xi_{1P} - \xi_{1N}\partial_M\xi_{2P})\partial_Q\chi \\ = -\frac{1}{4}\Gamma^M\Gamma^Q\Gamma^{NP}\partial_M(\xi_{2N}\xi_{1P})\partial_Q\chi - \frac{1}{4}\Gamma^M\Gamma^Q(\xi_2^N\partial_M\xi_{1N} - \xi_1^N\partial_M\xi_{2N})\partial_Q\chi. \end{aligned} \quad (\text{A.7})$$

The second term is of the form of a ξ^M gauge transformation. Commuting gamma matrices and using the weak constraint, the first term can be rewritten as a λ gauge transformation,

$$\begin{aligned} -\frac{1}{4}\Gamma^M\partial_M\left[(\Gamma^{NP}\Gamma^Q + 2(\eta^{QN}\Gamma^P - \eta^{QP}\Gamma^N))\xi_{2N}\xi_{1P}\partial_Q\chi\right] \\ = -\frac{1}{2}\not{\partial}\left[(\not{\xi}_2\not{\xi}_1 - \not{\xi}_1\not{\xi}_2)\not{\partial}\chi\right] - \frac{1}{2}\Gamma^M\partial_M\left[(\xi_2^N\Gamma^P\xi_{1P} - \xi_1^N\Gamma^P\xi_{2P})\partial_N\chi\right]. \end{aligned} \quad (\text{A.8})$$

The second term in here cancels against the the first term in (A.3).

Summarizing, the surviving structures are the ξ^M gauge transformations in (A.4) and (A.7) and the λ transformation in (A.8), combining into

$$[\delta_{\xi^1}, \delta_{\xi^2}]\chi = \frac{1}{2}\Gamma^P\Gamma^Q\left(\xi_2^M\partial_M\xi_{1P} - \frac{1}{2}\xi_2^M\partial_P\xi_{1M} - (1 \leftrightarrow 2)\right)\partial_Q\chi + \not{\partial}\lambda, \quad (\text{A.9})$$

with the λ parameter (2.38). Thus, the gauge algebra closes as stated in (2.36).

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